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Best Bounds for the Uniform Periodic Spline Interpolation Operator

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1. INTRODUCTION

Let $x_i = i/n$, $i = 0, 1,..., n$, be a uniform subdivision of $[0, 1]$ and k a positive integer. Then for each $f \in C[0, 1]$ satisfying $f(0) = f(1)$ there exists a unique periodic spline $\mathscr{L}_n^k f$ of degree k interpolating to f at the points $\{x_i\}_{i=0}^n$, having knots at these points or halfway between if k is, respectively, odd or even. We will be interested in describing the behavior of

$$\| \mathscr{L}_n^k \| = \sup_{\|f\|=1} \| \mathscr{L}_n^k f \|, \qquad n = 1, 2,...,$$

under the Chebyshev (sup) norm. This problem has been investigated by Schurer and Cheney [11] for cubics ($k = 3$) and by Schurer [10] for the quintic case ($k = 5$); in particular, formulas for $\| \mathscr{L}_n^k \|$ and $\sup_n \| \mathscr{L}_n^k \|$ are given. The same will be done here, in Section 4, for arbitrary degree $k \geqslant 2$. Of key importance is a very pleasant property of arbitrary periodic splines which will be described in Section 2.

A cardinal spline function of degree k is a spline having knots at the integers (half-integers) if k is odd (even). The bounded cardinal spline operator will be investigated in Section 5, where its intimate relation with the periodic spline operator will be noted.

I wish to express my appreciation to Professor I. J. Schoenberg for numerous helpful discussions and comments relating to the material presented here.

2. THE CYCLIC VARIATION DIMINISHING PROPERTY OF PERIODIC SPLINES

Let $x_1, x_2,..., x_r$ be a finite sequence of reals. Define $\nu(x_i)_{i=1}^r$ to be the number of variations of sign in the terms of the sequence, where terms $x_j = 0$ are not counted, and $\nu_c(x_i)_{i=1}^r$, the number of cyclic variations of the sequence, in the following manner. If $x_j \neq 0$, let

$$\nu_c(x_i)_{i=1}^r = \nu(x_j, x_{j+1}, ..., x_r, x_1, ..., x_j)$$

If all $x_i = 0$, we let $\nu_c(x_i) = \nu(x_i) = 0$. Note that $\nu_c(x_i)$ is an even number, in fact

$$\nu_c(x_i) = \begin{cases} \nu(x_i) & \text{if } \nu(x_i) \text{ is even,} \\ \nu(x_i) + 1 & \text{if } \nu(x_i) \text{ is odd.} \end{cases} \tag{2.1}$$

For a function f defined on $I = [0, 1]$, define

$$\nu(f) = \nu(f)_I = \sup \nu(f(t_i))_{i=1}^r$$

$$\nu_c(f) = \nu_c(f)_I = \sup \nu_c(f(t_i))_{i=1}^r$$

where the supremum is taken over all arbitrary finite sets $t_1, t_2, ..., t_r$ of increasing elements of I.

It has been shown [3] that spline functions possess the following variation diminishing property: Suppose $S(x)$ is a spline of degree k with knots

$$0 = x_0 < x_1 < \cdots < x_n = 1 \tag{2.2}$$

and representation

$$S(x) = \sum_{i=0}^{n-k-1} c_i M_i(x), \qquad x \in I,$$

where $M_i(x)$ is the i-th B-spline of degree k for the subdivision (2.2). Then

$$\nu(S)_I \leqslant \nu(c_i)_{i=0}^{n-k-1} \tag{2.3}$$

In considering a periodic spline on [0, 1], we first extend its knots in such a manner that all new knots are obtained from the old by translations of all integral sizes, i.e.,

$$x_{i+ln} = x_i + l \qquad i = 1, 2, ..., n; \quad l = 0, \pm 1, \pm 2, ... \tag{2.4}$$

Now define the periodic B-splines

$$\overline{M}_i(x) = \sum_{l=-\infty}^{\infty} M_{i+ln}(x) \qquad i = 1, 2, ..., n$$

Schoenberg [7] has shown that these functions form a basis for the space $\mathscr{S}_\pi^k$ of 1-periodic splines of degree k with knots (2.2).

Hence

$$S(x) = \sum_{i=1}^{n} c_i \overline{M}_i(x) \qquad x \in I. \tag{2.5}$$

Although (2.5) does not satisfy (2.3), we do have the following theorem

THEOREM 1. *Periodic splines are cyclic variation diminishing, i.e., if $S(x)$ is periodic of degree k with representation* (2.5), *then*

$$\nu_c(S)_I \leqslant \nu_c(c_i)_{i=1}^{n} \tag{2.6}$$

Proof. When considered as a function defined on $R = (-\infty, \infty)$, $S(x)$ clearly has knots (2.4). Let p be a positive integer to be chosen later. Restricting attention to the interval $[0, p]$, one may write

$$S(x) = \sum_{i=-k}^{pn-1} c_i M_i(x), \qquad x \in [0, p]$$

where

$$c_{i+n} = c_i\,, \qquad i = -k, -k+1, \dots, (p-1)n - 1 \tag{2.7}$$

and $c_1, c_2, \dots, c_n$ are the same as those in (2.5). From (2.3), we have

$$\begin{aligned} \nu(S)_{[0,p]} &\leqslant \nu(c_i)_{i=-k}^{pn-1} \\ &\leqslant \nu(c_i)_{i=-k}^{0} + \nu(c_i)_{i=0}^{pn-1} \end{aligned} \tag{2.8}$$

Also, (2.7) implies

$$\nu(c_i)_{i=0}^{pn-1} \leqslant p\nu_c(c_i)_{i=0}^{n-1} \tag{2.9}$$

If $\nu_c(S)_{[0,1]} = \nu_c(S(x_1), S(x_2), \dots, S(x_q))$, then by the periodicity of S and (2.1)

$$\begin{aligned} p\nu_c(S)_{[0,1]} &= \nu_c(S(x_1), \dots, S(x_q), S(x_1 + 1), \dots, S(x_q + 1), S(x_1 + 2), \\ &\qquad \dots, S(x_1 + (p-1)), \dots, S(x_q + (p-1))) \\ &\leqslant \nu(S)_{[0,p]} + 1, \end{aligned} \tag{2.10}$$

and hence by (2.8), (2.9), and (2.10)

$$p\nu_c(S)_{[0,1]} \leqslant \nu(c_i)_{i=-k}^{0} + 1 + p\nu_c(c_i)_{i=0}^{n-1} \tag{2.11}$$

Choosing $p \geqslant k + 2$ and since $\nu_c(\cdot)$ is an integer valued function, the theorem is proved.

Denote by $Z(S)$ the number of zeros of S on [0, 1] *not* counted according to multiplicitics. A zero at the endpoints is counted just once, as is an interval where $S(x) \equiv 0$. Noting that $\nu_c(S)$ is equal to the number of zeros through which S changes sign, and then using a simple variational argument, one may establish the following corollary.

COROLLARY 1. *Let S be a periodic spline on* [0, 1] *with knots* (2.2). *Then*

$$Z(S) \leqslant \begin{cases} n & \text{for } n \text{ even}, \\ n-1 & \text{for } n \text{ odd}. \end{cases} \tag{2.12}$$

Examples may be furnished to show the bounds are best possible. Also observe that the bounds are independent of the degree.

As a final remark, because of the paper of Jones and Karlovitz [2], we have the following curious fact.

COROLLARY 2. *Let n be odd in* (2.2). *Then any $f \in C[0, 1]$ has a best approximation $s \in \mathscr{P}_\pi^k$ where the error function $e = f - s$ equioscillates, i.e., there exist $n + 1$ points $0 \leqslant \xi_1 < \xi_2 < \cdots < \xi_{n+1} < 1$ such that*

$$e(\xi_i) = \| e \|$$

and

$$e(\xi_i)\, e(\xi_{i+1}) \leqslant 0.$$

On the other hand, this statement is *false* if n is even.

3. THE PERIODIC SPLINE OPERATOR

We now turn to the problem of computing the norm of the periodic spline operator on a uniform subdivision of [0, 1]. With no loss of generality, we may rescale so that $\mathscr{L}_n^k f$ has period n and interpolates to f at the integers. Let us also assume $n > 1$, as quite clearly $\| \mathscr{L}_1^k \| = 1$.

Define $L_n(x) = L_n^k(x)$ to be the cardinal spline of period n and degree k satisfying

$$L_n(\nu) = \begin{cases} 1 & \nu \equiv 0 \bmod n \\ 0 & \nu \not\equiv 0 \bmod n \end{cases} \qquad \nu = 0, \pm 1, \pm 2, \ldots . \tag{3.1}$$

Then clearly

$$\mathscr{L}_n^k f(x) = \sum_{\nu=1}^{n} f(\nu)\, L_n(x - \nu) \qquad -\infty < x < \infty. \tag{3.2}$$

Since

$$|\mathscr{L}_n^k f(x)| \leqslant \|f\|_\infty \sum_{\nu=1}^{n} |L_n(x-\nu)|$$

and $\sum_{\nu=1}^{n} |L_n(x-\nu)|$ has period 1, it follows that

$$\|\mathscr{L}_n^k\| = \max_{x\in[0,1]} \sum_{\nu=1}^{n} |L_n(x-\nu)|. \tag{3.3}$$

Hence if a sequence $\tilde{y}_\nu^n$ can be found such that for all $x \in [0, 1]$

$$\tilde{y}_\nu^n = \operatorname{sgn} L_n(x-\nu) \qquad \nu = 1, 2,..., n \tag{3.4}$$

then

$$\tilde{S}_n(x) = \sum_{\nu=1}^{n} \tilde{y}_\nu^n L_n(x-\nu)$$

will represent an extremal spline, i.e.,

$$\|\mathscr{L}_n^k\| = \max_{x\in[0,1]} \tilde{S}_n(x). \tag{3.5}$$

LEMMA 1. *Let $\tilde{y}_\nu^n$ be defined as follows:*

$$\begin{aligned} &\tilde{y}_\nu^n = -(-1)^\nu, \qquad \nu = 1, 2,..., n \quad \textit{for } n \textit{ odd};\\ &\tilde{y}_\nu^{2n} = \begin{cases} -(-1)^\nu, & \nu = 1, 2,..., n,\\ (-1)^\nu, & \nu = n+1,..., 2n. \end{cases} \end{aligned} \tag{3.6}$$

Then this definition satisfies the condition (3.4).

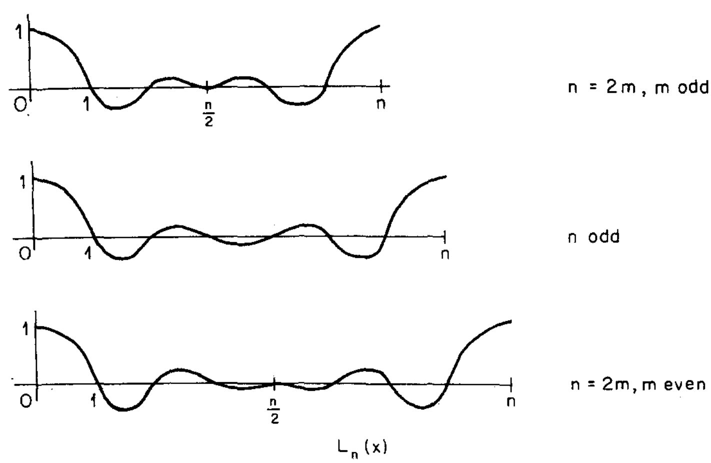


FIGURE 1

The lemma will be established by demonstrating that the splines $L_n(x)$ appear as in Fig. 1. Hence it will be sufficient to prove the following statements hold on $[0, n)$.

(a) $L_n(x)$ has zeros only at $x = 1, 2,..., n-1$;

(b) $L_n'(\nu) \neq 0$, $\nu = 1, 2,..., n-1$ for n odd;

(c) $L_n'(\nu) \neq 0$, $\nu = 1, 2,..., n-1$, $\nu \neq n/2$, n even.

Corollary 1 is used repeatedly throughout the proof.

Proof. (a) If n is odd, the statement is immediate. Suppose n even. Then by the unicity of cardinal spline interpolation, $L_n(x)$ is symmetric about $x = n/2$ (since the data (3.1) possesses this property). Thus if $L_n(x)$ had an additional zero on $(0, n)$, it would in fact have two, violating (2.12).

(b) Using Rolle's theorem and the periodicity of $L_n(x)$, we find that $L_n'(x)$ has exactly $n-1$ zeros on $[0, n)$, none of which is an integer (except possibly $x = 0$).

(c) By the same reasoning as in (b), $L_n'(x)$ has $n-1$ such zeros plus possibly one more, and this must occur at $x = n/2$ by the symmetry condition.

LEMMA 2.

$$\| \mathscr{L}_n^k \| = \tilde{S}_n(1/2). \tag{3.7}$$

Proof. We must show

$$\max_{x\in[0,1]} \tilde{S}_n(x) = \tilde{S}_n(1/2) \tag{3.8}$$

Assume n odd. By (3.6) and the intermediate value theorem, $\tilde{S}_n(x)$ has at least $n-1$ zeros on $(1, n)$, and hence $\tilde{S}_n'(x)$ has at least $n-2$ zeros on this interval. Thus $\tilde{S}_n'(x)$ may have at most one zero on $[0, 1]$, which we claim occurs at $x = 1/2$. To see this, extend the data $\{\tilde{y}_\nu^n\}_{\nu=1}^n$ periodically, i.e., let

$$y_\nu^{2n} = \begin{cases} -(-1)^\nu & \nu = 2ln+1, 2ln+2,..., 2ln+n \\ (-1)^\nu & \nu = (2l+1)n+1, (2l+1)n+2,..., (2l+1)n+n, \end{cases}$$
$$l = 0, \pm 1, \pm 2,...,$$

$$y_\nu^n = y_\nu^{2n} \qquad \text{for} \quad n \text{ odd}, \tag{3.9}$$

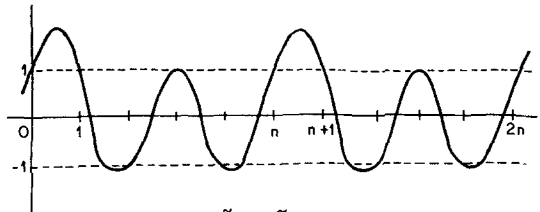
where

$$y_{\nu+ln}^n = y_\nu^n = \tilde{y}_\nu^n \qquad \nu = 1, 2,..., n, \quad l = 0, \pm 1, \pm 2,... .$$

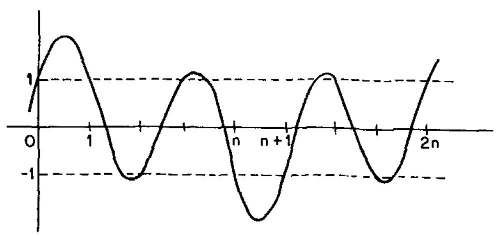
Clearly,

$$\tilde{S}_n(\nu) = y_\nu^n, \qquad \nu = 0, \pm 1, \pm 2,... .$$

Since the data (3.9) is symmetric about $x = 1/2$, so is $\tilde{S}_n(x)$. Hence $\tilde{S}_n'(1/2) = 0$, and it can be shown that this is indeed a maximum for $\tilde{S}_n(x)$. The proof for n even is similar.



$\tilde{S}_n(x) = \tilde{S}_{2n}(x)$, n odd



$\tilde{S}_{2n}(x)$, n even

FIGURE 2.

4. CONSTRUCTION OF $\tilde{S}_n(x)$

In this section, it will be assumed that our splines have odd degree, i.e., $k = 2m - 1$, and hence all knots are located at the integers. Proofs for the even degree case are quite similar, and the corresponding results given later.

Define $\mathscr{S}_{2m-1}$ to be the set of cardinal splines of degree $2m - 1$ vanishing at all integers. Schoenberg [8] has shown that $\mathscr{S}_{2m-1}$ is a linear space of $2m - 2$ dimensions spanned by "eigensplines" $S_1(x), S_2(x),..., S_{2m-2}(x)$. Corresponding to this basis is a set of reals

$$\lambda_1 < \lambda_2 < \cdots < \lambda_{m-1} < -1 < \lambda_m < \cdots < \lambda_{2m-2} < 0, \qquad (4.1)$$

such that

$$S_i(x + 1) = \lambda_i S_i(x) \qquad i = 1, 2,..., 2m - 2. \qquad (4.2)$$

We will also need the Euler spline, $E(x)$, which is the unique bounded cardinal spline of degree $2m - 1$ satisfying

$$E(\nu) = (-1)^\nu \qquad \nu = 0, \pm 1, \pm 2, \ldots . \tag{4.3}$$

Consider the restriction of $\tilde{S}_{2n}(x)$ to the interval $[-n + 1, 0]$. This function may be uniquely extended to a cardinal spline $\bar{S}_{2n}(x)$ interpolating to the Euler data (4.3). Therefore $\bar{S}_{2n} - E \in \mathscr{S}_{2m-2}$ and so there exist real numbers $c_1, c_2, \ldots, c_{2m-2}$ such that

$$\bar{S}_{2n}(x) = E(x) + \sum_{i=1}^{2m-2} c_i S_i(x), \qquad -\infty < x < \infty,$$

or

$$\tilde{S}_{2n}(x) = E(x) + \sum_{i=1}^{2m-2} c_i S_i(x), \qquad x \in [-n + 1, 0]. \tag{4.4}$$

If n is odd, the data (3.9) (and hence $\tilde{S}_n(x)$) is symmetric about $x = 1/2$ and $x = (-n + 1)/2$. Since $\tilde{S}_n \in C^{2m-2}(R)$ and $x = 1/2$ is *not* a knot, we have the relations

$$\begin{aligned} \tilde{S}_n^{(\nu)}(1/2) &= 0, \qquad \nu = 1, 3, \ldots, 2m - 1, \\ \tilde{S}_n^{(\nu)}\left(\frac{-n + 1}{2}\right) &= 0, \qquad \nu = 1, 3, \ldots, 2m - 3. \end{aligned} \tag{4.5}$$

Letting

$$t_+ = \begin{cases} t, & t \geqslant 0; \\ 0, & t < 0. \end{cases}$$

we may represent $\tilde{S}_n(x)$ on $[-n + 1, 1]$ in the form

$$\tilde{S}_n(x) = E(x) + \sum_{i=1}^{2m-2} c_i S_i(x) + \frac{a}{(2m - 1)!} x_+^{2m-1}, \qquad x \in [-n + 1, 1], \tag{4.6}$$

and the unknowns may be obtained by applying (4.5) to yield the nonsingular system of equations

$$\begin{aligned} E^{(\nu)}\left(\frac{1}{2}\right) + \sum_{i=1}^{2m-2} c_i S_i^{(\nu)}\left(\frac{1}{2}\right) + \frac{a}{(2m - 1 - \nu)!}\left(\frac{1}{2}\right)^{2m-1-\nu} &= 0, \\ \nu = 1, 3, \ldots, 2m - 1&, \\ \sum_{i=1}^{2m-2} c_i \lambda_i^{-((n-1)/2)} S_i^{(\nu)}(0) &= 0, \\ \nu = 1, 3, \ldots, 2m - 3&. \end{aligned} \tag{4.7}$$

The last $m-1$ equations follow from the quasiperiodicity property (4.2) and the evenness of $E(x)$ about all integers.

For n even, we first observe that $\tilde{S}_{2n}(x)$ is symmetric about $x = 1/2$ and $x = -n + 1/2$, and since neither of these points are knots, it follows that

$$\tilde{S}_{2n}^{(\nu)}(1/2) = \tilde{S}_{2n}^{(\nu)}(-n+1/2) = 0, \qquad \nu = 1, 3,\ldots, 2m-1. \tag{4.8}$$

Also

$$\tilde{S}_{2n}(x) = E(x) + \sum_{i=1}^{2m-2} c_i S_i(x) + \frac{a}{(2m-1)!} x_+^{2m-1} + \frac{b}{(2m-1)!}(-n+1-x)_+^{2m-1}, \qquad x \in [-n, 1]. \tag{4.9}$$

Hence,

$$\begin{gathered} E^{(\nu)}\left(\frac{1}{2}\right) + \sum_{i=1}^{2m-2} c_i S_i^{(\nu)}\left(\frac{1}{2}\right) + \frac{a}{(2m-1-\nu)!}\left(\frac{1}{2}\right)^{2m-1-\nu} = 0 \\ E^{(\nu)}\left(\frac{1}{2}\right) + \sum_{i=1}^{2m-2} c_i \lambda_i^{-n} S_i^{(\nu)}\left(\frac{1}{2}\right) - \frac{b}{(2m-1-\nu)!}\left(\frac{1}{2}\right)^{2m-1-\nu} = 0, \\ \nu = 1, 3,\ldots, 2m-1. \end{gathered} \tag{4.10}$$

On solving (4.7) and (4.10) and making use of Lemma 2, we have established the following theorem.

THEOREM 2. *For n odd, $n > 1$,*

$$\| \mathscr{L}_n^{2m-1} \| = \| \mathscr{L}_{2n}^{2m-1} \|$$

$$= \begin{vmatrix} E\left(\frac{1}{2}\right) & S_1\left(\frac{1}{2}\right) & \cdots & S_{2m-2}\left(\frac{1}{2}\right) & \frac{1}{(2m-1)!}\left(\frac{1}{2}\right)^{2m-1} \\ E'\left(\frac{1}{2}\right) & S_1'\left(\frac{1}{2}\right) & \cdots & S_{2m-2}'\left(\frac{1}{2}\right) & \frac{1}{(2m-2)!}\left(\frac{1}{2}\right)^{2m-2} \\ \vdots & \vdots & & \vdots & \vdots \\ E^{(2m-1)}\left(\frac{1}{2}\right) & S_1^{(2m-1)}\left(\frac{1}{2}\right) & \cdots & S_{2m-2}^{(2m-1)}\left(\frac{1}{2}\right) & 1 \\ 0 & \lambda_1^{-((n-1)/2)} S_1'(0) & \cdots & \lambda_{2m-2}^{-((n-1)/2)} S_{2m-2}'(0) & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \lambda_1^{-((n-1)/2)} S_1^{(2m-3)}(0) & \cdots & \lambda_{2m-2}^{-((n-1)/2)} S_{2m-2}^{(2m-3)}(0) & 0 \end{vmatrix} \cdot \Delta_n^{-1}. \tag{4.11}$$

For even n,

$$\| \mathscr{L}_{2n}^{2m-1} \| = \begin{vmatrix} E\left(\frac{1}{2}\right) & S_1\left(\frac{1}{2}\right) & \cdots & S_{2m-2}\left(\frac{1}{2}\right) & \frac{1}{(2m-1)!}\left(\frac{1}{2}\right)^{2m-1} & 0 \\ E'\left(\frac{1}{2}\right) & S_1'\left(\frac{1}{2}\right) & \cdots & S_{2m-2}'\left(\frac{1}{2}\right) & \frac{1}{(2m-2)!}\left(\frac{1}{2}\right)^{2m-2} & \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ E^{(2m-1)}\left(\frac{1}{2}\right) & S_1^{(2m-1)}\left(\frac{1}{2}\right) & \cdots & S_{2m-2}^{(2m-1)}\left(\frac{1}{2}\right) & 1 & 0 \\ E'\left(\frac{1}{2}\right) & \lambda_1^{-n}S_1'\left(\frac{1}{2}\right) & \cdots & \lambda_{2m-2}^{-n}S_{2m-2}'\left(\frac{1}{2}\right) & 0 & -\frac{1}{(2m-2)!}\left(\frac{1}{2}\right)^{2m-2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ E^{(2m-1)}\left(\frac{1}{2}\right) & \lambda_1^{-n}S_1^{(2m-1)}\left(\frac{1}{2}\right) & \cdots & \lambda_{2m-2}^{-n}S_{2m-2}^{(2m-1)}\left(\frac{1}{2}\right) & 0 & -1 \end{vmatrix} \cdot \Delta_n^{-1}. \tag{4.12}$$

Δ_n is the minor of the leading term $E(1/2)$.

5. The Cardinal Spline Operator

For each function $f(x)$ bounded on R, we define $\mathscr{L}^k f$ to be the unique bounded cardinal spline of degree k satisfying

$$\mathscr{L}^k f(\nu) = f(\nu) \qquad \nu = 0, \pm 1, \pm 2, \ldots .$$

It should be noted that one of the main tools used to investigate periodic splines has been to consider them as cardinal splines. Thus our methods should enable us to furnish a value for $\| \mathscr{L}^k \|$.

Proceeding as before, we find that

$$\| \mathscr{L}^k \| = \max_{x \in [0,1]} \sum_{\nu=-\infty}^{\infty} | L(x - \nu)|$$

where $L(x)$ is the unique bounded cardinal spline of degree k interpolating to the data

$$L(\nu) = \begin{cases} 1, & \nu = 0 \\ 0, & \nu = \pm 1, \pm 2, \ldots . \end{cases}$$

It is known that

$$\operatorname*{sgn}_{x \in [0,1]} L(x - \nu) = y_\nu = \begin{cases} -(-1)^\nu, & \nu = 1, 2, \ldots, \\ (-1)^\nu, & \nu = 0, -1, -2, \ldots . \end{cases} \tag{5.1}$$

Hence, letting

$$\tilde{S}(x) = \sum_{\nu=-\infty}^{\infty} y_\nu L(x-\nu),$$

we have

$$\| \mathscr{L}^k \| = \max_{x\in[0,1]} \tilde{S}(x) \tag{5.2}$$

The relationship between $\| \mathscr{L}^k \|$ and $\| \mathscr{L}_n{}^k \|$ is expressed by the following theorem.

THEOREM 3.

$$\lim_{n\to\infty} \| \mathscr{L}_n{}^k \| = \| \mathscr{L}^k \| = \sup_n \| \mathscr{L}_n{}^k \|. \tag{5.3}$$

Proof. Restricting attention to [0, 1],

$$| \tilde{S}(x) - \tilde{S}_n(x)| \leqslant \sum_{\nu=-\infty}^{\infty} | y_\nu - y_\nu{}^n | \, | L(x-\nu)|$$

$$\leqslant 2 \sum_{|\nu|\geqslant n/2} | L(x-\nu)|, \qquad x \in [0, 1],$$

and since $L(x) \to 0$ exponentially as $| x | \to \infty$

$$\tilde{S}_n(x) \to \tilde{S}(x) \qquad \text{as} \qquad n \to \infty \tag{5.4}$$

uniformly on [0, 1]. Then by (3.8) and (5.2)

$$\| \mathscr{L}^k \| = \tilde{S}(1/2). \tag{5.5}$$

The theorem follows by setting $x = 1/2$ in (5.4). Note the second equality in (5.3) holds because of the obvious fact

$$\| \mathscr{L}^k \| \geqslant \| \mathscr{L}_n{}^k \|.$$

Let us now proceed with the computation of $\| \mathscr{L}^{2m-1} \|$. Extend $\tilde{S}(x)$ from $(-\infty, 0]$ to R as in Section 4, i.e., the extension, $\bar{S}(x)$, is a cardinal spline satisfying the Euler data (4.3). Then we may write

$$\bar{S}(x) - E(x) = \sum_{i=1}^{2m-2} c_i S_i(x). \tag{5.6}$$

However, both $\bar{S}(x)$ and $E(x)$ are *bounded* for $x \leqslant 0$. Therefore only those eigensplines $S_i(x)$ which are bounded for $x \leqslant 0$ may appear in (5.6), and because of (4.2) we find

$$\bar{S}(x) - E(x) = \sum_{i=1}^{m-1} c_i S_i(x), \qquad -\infty < x < \infty,$$

and hence

$$\tilde{S}(x) = E(x) + \sum_{i=1}^{m-1} c_i S_i(x) + \frac{a}{(2m-1)!} x_+^{2m-1}, \qquad x \leqslant 1. \tag{5.7}$$

Because the data (5.1) is symmetric about $x = 1/2$, one obtains the equations

$$\tilde{S}^{(\nu)}(1/2) = 0, \qquad \nu = 1, 3, \ldots, 2m-1 \tag{5.8}$$

enabling us to solve for the unknowns in (5.7). This leads to the following expression for $\| \mathscr{L}^{2m-1} \|$.

THEOREM 4.

$$\| \mathscr{L}^{2m-1} \|$$
$$= \begin{vmatrix} E\left(\frac{1}{2}\right) & S_1\left(\frac{1}{2}\right) & \cdots & S_{m-1}\left(\frac{1}{2}\right) & \frac{1}{(2m-1)!}\left(\frac{1}{2}\right)^{2n-1} \\ E'\left(\frac{1}{2}\right) & S_1{}'\left(\frac{1}{2}\right) & \cdots & S'_{m-1}\left(\frac{1}{2}\right) & \frac{1}{(2m-2)!}\left(\frac{1}{2}\right)^{2m-2} \\ E'''\left(\frac{1}{2}\right) & S_1'''\left(\frac{1}{2}\right) & \cdots & S'''_{m-1}\left(\frac{1}{2}\right) & \frac{1}{(2m-4)!}\left(\frac{1}{2}\right)^{2m-.} \\ \vdots & \vdots & & \vdots & \\ E^{(2m-1)}\left(\frac{1}{2}\right) & S_1^{(2m-1)}\left(\frac{1}{2}\right) & \cdots & S_{m-1}^{(2m-1)}\left(\frac{1}{2}\right) & 1 \end{vmatrix} \cdot \Delta^{-1}, \tag{5.9}$$

where Δ is the minor of the leading element $E(1/2)$.

We now give the results for even degree $k = 2m$. The functions $E(x)$, $S_i(x)$ are as defined before, except now their knots are at the half-integers. We also note that $\mathscr{L}_{2m}$ has dimension $2m$, and the corresponding eigenvalues satisfy $\lambda_1 < \lambda_2 < \cdots < \lambda_m < -1 < \lambda_{m+1} < \cdots < \lambda_{2m} < 0$. Δ_n and Δ have meanings as before.

THEOREM 5. *For n odd, $n > 1$,*

$$\| \mathscr{L}_n^{2m} \| = \| \mathscr{L}_{2n}^{2m} \|$$

$$= \begin{vmatrix} E\left(\frac{1}{2}\right) & S_1\left(\frac{1}{2}\right) & \cdots & S_{2m}\left(\frac{1}{2}\right) \\ E'\left(\frac{1}{2}\right) & S_1'\left(\frac{1}{2}\right) & & S_{2m}'\left(\frac{1}{2}\right) \\ \vdots & \vdots & & \vdots \\ E^{(2m-1)}\left(\frac{1}{2}\right) & S_1^{(2m-1)}\left(\frac{1}{2}\right) & & S_{2m}^{(2m-1)}\left(\frac{1}{2}\right) \\ 0 & \lambda_1^{-((n-1)/2)} S_1'(0) & & \lambda_{2m}^{-((n-1)/2)} S_{2m}'(0) \\ \vdots & \vdots & & \vdots \\ 0 & \lambda_1^{-((n-1)/2)} S_1^{(2m-1)}(0) & \cdots & \lambda_{2m}^{-((n-1)/2)} S_{2m}^{(2m-1)}(0) \end{vmatrix} \cdot \Delta_n^{-1}. \tag{5.10}$$

For n even,

$$\| \mathscr{L}_{2n}^{2m} \| = \begin{vmatrix} E\left(\frac{1}{2}\right) & S_1\left(\frac{1}{2}\right) & \cdots & S_{2m}\left(\frac{1}{2}\right) \\ E'\left(\frac{1}{2}\right) & S_1'\left(\frac{1}{2}\right) & & S_{2m}'\left(\frac{1}{2}\right) \\ \vdots & \vdots & & \vdots \\ E^{(2m-1)}\left(\frac{1}{2}\right) & S_1^{(2m-1)}\left(\frac{1}{2}\right) & & S_{2m}^{(2m-1)}\left(\frac{1}{2}\right) \\ E'\left(\frac{1}{2}\right) & \lambda_1^{-n} S_1'\left(\frac{1}{2}\right) & & \lambda_{2m}^{-n} S_{2m}'\left(\frac{1}{2}\right) \\ \vdots & \vdots & & \vdots \\ E^{(2m-1)}\left(\frac{1}{2}\right) & \lambda_1^{-n} S_1^{(2m-1)}\left(\frac{1}{2}\right) & \cdots & \lambda_{2m}^{-n} S_{2m}^{(2m-1)}\left(\frac{1}{2}\right) \end{vmatrix} \cdot \Delta_n^{-1}. \tag{5.11}$$

$$\| \mathscr{L}^{2m} \| = \begin{vmatrix} E\left(\frac{1}{2}\right) & S_1\left(\frac{1}{2}\right) & \cdots & S_m\left(\frac{1}{2}\right) \\ E'\left(\frac{1}{2}\right) & S_1'\left(\frac{1}{2}\right) & \cdots & S_m'\left(\frac{1}{2}\right) \\ \vdots & \vdots & & \vdots \\ E^{(2m-1)}\left(\frac{1}{2}\right) & S_1^{(2m-1)}\left(\frac{1}{2}\right) & \cdots & S_m^{(2m-1)}\left(\frac{1}{2}\right) \end{vmatrix} \cdot \Delta^{-1}. \tag{5.12}$$

6. Other Remarks

To apply the formulas of the preceding sections, it is of course necessary to compute the Euler spline and all eigensplines. A few words on their construction seems in order. We assume the degree is odd.

It is not hard to see that the Euler spline on [0, 1] is that polynomial of degree $2m - 1$ satisfying

$$\begin{gathered} E(0) = 1, \\ E(1) = -1, \\ E^{(\nu)}(0) = E^{(\nu)}(1) = 0, \qquad \nu = 1, 3, \ldots, 2m-3. \end{gathered} \tag{6.1}$$

The eigenspline $S_i(x)$ is an element of $\mathscr{L}_{2m-1}$, satisfies (4.2), and is of class C^{2m-2}. Therefore,

$$\begin{aligned} S_i(0) &= S_i(1) = 0, \\ S_i^{(\nu)}(1) &= \lambda_i S_i^{(\nu)}(0), \qquad \nu = 1, 2, \ldots, 2m-2. \end{aligned} \tag{6.2}$$

Since $S_i(x)$ is just a polynomial on [0, 1], (6.2) forms a homogeneous system of equations from which the eigenvalues and corresponding eigensplines may be obtained. We illustrate for the cubic case. Writing

$$S_i(x) = a_1 x^3 + 3a_2 x^2 + 3a_3 x, \qquad x \in [0, 1],$$

we apply (6.2) and find

$$\begin{aligned} a_1 + a_2 &= \lambda a_2 \\ a_1 + 2a_2 + a_3 &= \lambda a_3 \\ a_1 + 3a_2 + 3a_3 &= 0. \end{aligned}$$

This gives the characteristic equation

$$\begin{vmatrix} 1 & 1-\lambda & 0 \\ 1 & 2 & 1-\lambda \\ 1 & 3 & 3 \end{vmatrix} = \lambda^2 + 4\lambda + 1 = 0,$$

whence,

$$\begin{aligned} \lambda_1 &= -2 - \sqrt{3}, \\ \lambda_2 &= -2 + \sqrt{3}. \end{aligned}$$

We may now solve for $S_i(x)$:

$$\begin{aligned} S_1(x) &= -(3 + \sqrt{3})\, x^3 + 3x^2 + \sqrt{3}\, x, \\ S_2(x) &= (-3 + \sqrt{3})\, x^3 + 3x^2 - \sqrt{3}\, x, \end{aligned}$$

Also,

$$E(x) = 4x^3 - 6x^2 + 1.$$

Formulas (4.11), (4.12), and (5.9) may now be applied. In particular

$$\| \mathscr{L}^3 \| = (1 + 3\sqrt{3})/4 \approx 1.55.$$

Other computational results of interest are as follows:

$$\| \mathscr{L}^2 \| = \sqrt{2} \approx 1.41,$$
$$\| \mathscr{L}^4 \| \approx 1.69,$$
$$\| \mathscr{L}^5 \| \approx 1.82.$$

We conclude the paper with an approximation theorem. The following modification of a result of Marsden [5] is used.

LEMMA (Marsden). *There exists a linear operator T from $C[0, 1]$ onto the space of splines of degree k with knots* (2.2) *having the property*

$$\|f - Tf\| \leqslant \left(\left(\frac{k+1}{12}\right)^{1/2} + 1\right) \omega(f; h), \tag{6.3}$$

where $\omega(f; \cdot)$ is the modulus of continuity of $f(x)$ and h is the mesh length of the subdivision (2.2). *In addition, if $f(0) = f(1)$, Tf is periodic.*

We now assume (2.2) is uniform and call the corresponding operator T_n^k. If $f(0) = f(1)$, then

$$\begin{aligned} \|f - \mathscr{L}_n^k f\| &\leqslant \|(f - T_n^k f) + \mathscr{L}_n^k(T_n^k f - f)\| \\ &\leqslant (1 + \| \mathscr{L}_n^k \|) \|f - T_n^k f\| \\ &\leqslant (1 + \| \mathscr{L}^k \|) \left(\left(\frac{k+1}{12}\right)^{1/2} + 1\right) \omega\left(f; \frac{1}{n}\right). \end{aligned} \tag{6.4}$$

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